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Sketching Rational Functions

Rational functions are of the form: $\frac{P(x)}{Q(x)}$. To graph without using a graphing calculator, we need to find the x & y intercepts, Vertical asymptotes, Horizontal asymptotes, and a few sample values.

There are a few standard steps that will help us assemble information about our graph.

1. Factor both the top and the bottom.
2. Find the x-intercept (set the top equal to 0).
3. Find the y-intercept. (find the value of y when the x =0).
4. Find the vertical asymptotes (set all factors of the bottom equal to zero) These will give us vertical lines.
5. Find the horizontal asymptotes using the following patterns: $\frac{a_1x^n + a_2x^{n-1} + \cdots .. a_0}{b_1x^m + b_2x^{m-1} + \cdots .. b_0}$
   a. If n < m, then the function has horizontal asymptote y = 0.
   b. If n = m, then the function has horizontal asymptotes y = $\frac{a_1}{b_1}$.
   c. If n > m, then the function has no horizontal asymptotes.
   d. If n = m +1, the graph has a slant asymptote. The equation of this may be found by dividing the top by the bottom.
6. Using the information gathered from the previous steps, graph as much as possible. You may need to plot a few additional points to gather the information needed to fill in the rest of the graph. This is intended to give a sketch not necessarily a detailed graph.

Example 1:

h(x) = $\mathbf{\frac{x^2-x-6}{x^2+3x}}$

1. Factor the top and the bottom: $\frac{(x-3)(x+2)}{x(x+3)}$

2. Find any x-intercepts by setting the top equal to 0:

   $(x - 3)(x + 2) = 0$

   $x - 3 = 0 \ \& \ x + 2 = 0$

   $x = 3 \ \& \ x = -2$

   $(3,0)$& $(-2,0)$: x- intercepts

3. Find any y intercepts by setting all the x's to 0:

   $\frac{(0-3)(0+2)}{0(0+3)}$

   $\frac{-6}{0}$ undefined, no y intercept

4. Find any vertical asymptotes by setting the bottom to 0:

   $x = 0 \ \& \ x + 3 = 0$

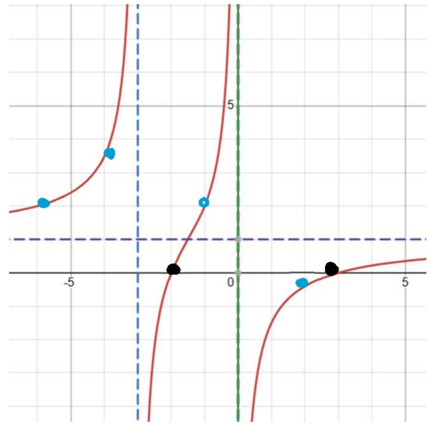
   $x = 0 \ \& \ x = -3$

5. Find any horizontal asymptotes by comparing the degrees:

   $\frac{x^2-x-6}{x^2+3x}$

   Degree top= 2
   Degree bottom = 2
   Horizontal asymptote: y = 1

6. Graph the previous information:
   Pts: (3,0) & (-2,0)
   Vertical asymptotes: x = 0 & x = -3
   Horizontal asymptotes: y = 1
   Finding a few more points
   (from the original equation)



h(x) = $\frac{x^2-x-6}{x^2+3x}$

x	y
-4	3.5
-6	2
-1	2
2	-.4

Now, sketch.

End behaviors:

As x -> -3 from the left, y -> +∞

As x -> -3 from the right, y -> -∞

As x -> 0 from the left, y -> +∞

As x -> 0 from the right, y -> -∞

Example 2:

$$G(x) = \frac{3(x^2+2)}{x^2-2x-3}$$

7. Factor the top and the bottom: $\frac{3(x^2+2)}{(x-3)(x+1)}$

8. Find any x-intercepts by setting the top equal to 0: $3(x^2+6) = 0$
   $x^2 = -6$
   No real answer: so, no x- intercepts

9. Find any y intercepts by setting all the x's to 0: $\frac{3(0^2+2)}{(0-3)(0+1)}$
   $\frac{6}{-2}$ = -3, (0,-2) y-intercept

10. Find any vertical asymptotes by setting the bottom to 0: $x - 3 = 0 \,\&\, x + 1 = 0$
    $x = 3 \,\&\, x = -1$

11. Find any horizontal asymptotes by comparing the degrees: $\frac{3(x^2+2)}{x^2-2x-3}$

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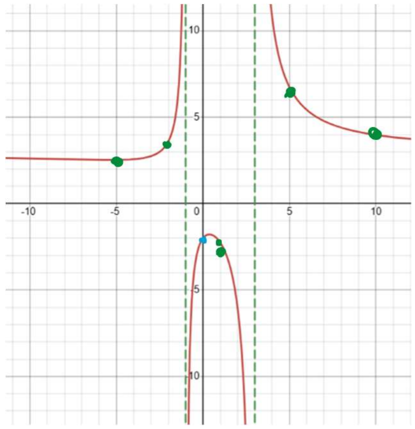
12. $G(x) = \frac{3(x^2+2)}{x^2-2x-3}$

x	y
1	-2
-2	3.6
-5	2.5
5	6.75
10	4



Degree top= 2
Degree bottom = 2
Horizontal asymptote: y = 3

End behaviors:

As x -> 3 from the left, y -> -∞

As x -> 3 from the right, y -> ∞

As x -> -1 from the left, y -> +∞

As x -> -1 from the right, y -> -∞

Example 3: $s(x) = \frac{x^2+2x}{x-1}$

1. Factor the top and the bottom: $\frac{x(x+2)}{x-1}$

2. Find any x-intercepts by setting the top equal to 0:
   $x(x+2) = 0$
   $x = 0 \,\&\, x = -2$
   $(0,0)\&(-2,0)$; x- intercepts

3. Find any y intercepts by setting all the x's to 0:
   $\frac{0(0+0)}{(0-1)}$
   $\frac{0}{-1} = 0, (0,0)$; y intercept

4. Find any vertical asymptotes by setting the bottom to 0:
   $x - 1 = 0$
   $x = 1$

5. Find any horizontal asymptotes by comparing the degrees:
   $\frac{x^2+2x}{x-1}$

Degree top = 2 Degree bottom = 1
This gives us a slant asymptote.

$x + 3$
$x - 1 \;\; \sqrt{x^2 + 2x}$
$\underline{-x^2 + x}$
$3x$
$\underline{-3x + 3}$
3

Equation of slant Asymptote: y = x + 3

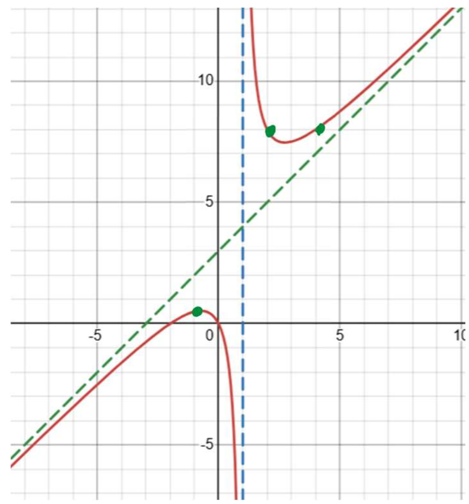
6. Graph the previous information:
   Pts: (0,0) & (-2,0)
   Vertical asymptotes: x = 1
   Slant asymptotes: y = x + 3
   Finding a few more points
   (from the original equation)

x	y
2	8
4	8
-1	.5

End behaviors:

As x -> 1 from the left, y -> -∞

As x -> 1 from the right, y -> ∞



Hint: how to determine end behaviors

To determine the behavior coming from the left, use a number slightly less (-0.1) than the asymptote number. When substituted into the original function, do you get a positive or negative number? If you get a positive number, then as you approach from the left, the y → +∞. If the value was negative, then the y → - ∞. To determine the behavior coming from the right, do the same with a number slightly greater (+0.1) than the asymptote number.