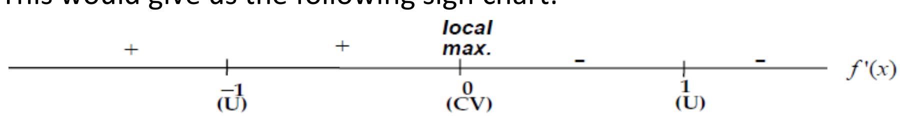
Graphing with Calculus

The Graphing Steps:

1. Find domain of $f(x)$.
2. Find *the first derivative of* $f(x)$.
3. Find **partition points (critical values and undefined values)** as follows:
    a. Set *the first derivative of* $f(x) = 0$ to find critical values.
    b. Set the denominator of *the first derivative of* $f(x) = 0$ to find undefined points.
4. Use ***either*** **First or Second Derivative Test** to determine local maxima & minima (*extrema*). Set the second derivative of $f(x) = 0$ to find critical values. Plot extrema.
5. Find other points to sketch the graph, including: (a) **intercepts**; (b) **inflection points** (second derivative sign changes) (c) **asymptotes** (see reverse side); (d) **concavity** and (e) **maximum/minimum values**.

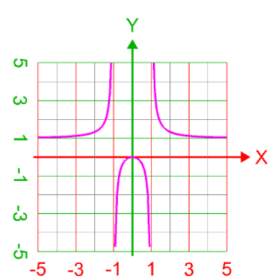
Problem 1. Use the **first derivative test** to sketch the graph of $f(x) = \frac{x^2}{x^2 - 1}$.

1. Find the **domain**. Set the denominator of $f(x)$ equal to zero, the domain is: $x \neq 1$ and $x \neq -1$.
2. Find the **first derivative** of $f(x)$. $f'(x) = \frac{-2x}{(x^2-1)^2}$
3. **Critical Values**:
    a. Set the first derivative = 0 (top of $f(x) = 0$). This yields x = 0. Thus 0 is a **critical value**.
    b. Set the denominator of $f(x) = 0$. This yields $x = 1$ and $x = -1$. Thus, both are critical values and will give **vertical asymptotes** on the graph.
4. **Local Extrema**: Use the **first derivative test.**
    a. Make a sign chart using partition values. On the chart, use a test point in $f'(x)$ on each interval between partition numbers. This is a good time to use the table function (set to ask) on the calculator.
        i. If $f'(x) > 0$, write $+$ on the sign chart, this is where $f(x)$ is **increasing**.
        ii. If $f'(x) < 0$, write $-$ on the sign chart, this is where $f(x)$ is **decreasing**.
        iii. If the sign changes from $(-)$ to $(+)$ you have a **local minimum**
        iv. If the sign changes from $(+)$ to $(-)$ you have a **local maximum**
    b. This would give us the following sign chart:



5. Now it is time to sketch our function. Since $\lim_{x \to \pm\infty} f(x) = 1$, the line $y = 1$ is a **horizontal asymptote**. We have two vertical asymptotes from step #3. Plotting other points as necessary completes the graph.

x	*f*(*x*)
-3	9/8
9/10	-81/19
0	0
4	-81/19
2	4/3
3	9/8



Problem #2. Use the **second derivative test** to graph $f(x) = x^3 - 6x^2$, on the interval $[-3,5]$, and find the absolute extrema.

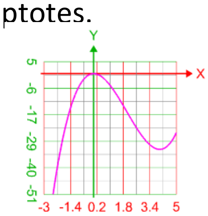
1. Determine the **domain**. In this case $(-\infty, \infty)$ since we don't have a fraction with domain limits.
2. Calculate the **first derivative**: $f'(x) = 3x^2 - 12x$.
3. **Critical Values**: Set $f'(x) = 0$, this yields x = 0 and 4. Thus, these are the **critical values**. Since this is not a function with a limited domain, there are no undefined points or vertical asymptotes.
4. **Relative Extrema**: Use the second derivative test. $f''(x) = 6x - 12$. Find $f''(0)$ and $f''(4)$.
   a. Since $f''(0) = -12 < 0, f(0)$ is a **local maximum**.
   b. Since $f''(4) = 12 > 0, f(4)$ is a **local minimum**.

5. **Concavity/ Inflection Points:** Setting $f''(x) = 0$ yields $x = 2$, so $f(x)$ has an **inflection point** at $(2, -16)$ and the graph changes concavity at $x = 2$.
   a. Since $f''(x) < 0$ on $(-\infty, 2)$, this part of the graph is **concave down**.
   b. Since $f''(x) > 0$ on $(2, \infty)$, this part of the graph is **concave up**.

6. **Absolute Extrema:** We know that $f(0)$ is a local maximum and that $f(0) = 0$ and $f(4)$ is a local minimum and $f(4) = -32$, so now we need to check the endpoints. $f(-3) = -81$ and $f(5) = -25$.
   a. $f(0) = 0$ which is the largest value so it is an **absolute maximum**.
   b. $f(-3) = -81$ which is the smallest so it is an **absolute minimum**.

7. Now it is time to sketch the graph. Using the critical values you can get a good idea of what the function looks like. This graph would not have any vertical or horizontal asymptotes.

x	$f(x)$
-3	-81
0	0
4	-32
5	-25



<u>Second Derivative Test</u>

A. Find $f''(x)$. B. <u>Check all CV's</u>: (i) if $f''(\text{CV}) > 0$, then $f(\text{CV})$ is a ***local minimum*** (ii) if $f''(\text{CV}) < 0$, then $f(\text{CV})$ is a ***local maximum*** (iii) if $f''(\text{CV}) = 0$, then test fails (use First Derivative Test)	$f''(\text{CV}) > 0$  *(local min./concave up)*	$f''(\text{CV}) < 0$  *(local max./concave down)*

<u>Notes on Asymptotes</u>:

1. The line $x = c$ is a ***<u>vertical asymptote</u>*** of the graph of $f(x) = \dfrac{p(x)}{q(x)}$ if $q(c) = 0$ and $p(c) \neq 0$.
2. The line $y = \lim\limits_{x \to \pm\infty} f(x) = b$ is a ***<u>horizontal asymptote</u>*** of $f(x)$ if b is a constant.
   If $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomials, and if $p(x)$ and $q(x)$ have the *<u>same degree</u>*, then the graph of $f(x)$ has a horizontal asymptote.
3. The line $y = ax + b$ is an ***<u>oblique asymptote</u>*** of the graph of $f(x)$ if $\lim\limits_{x \to \pm\infty} (f(x) - y) = 0$.
   If $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are both polynomials, and if the <u>degree</u> of $p(x)$ is <u>one more than</u> the degree of $q(x)$, then the graph of $f(x)$ has an oblique asymptote. The equation of this asymptote is $y = Q(x) = ax + b$, where $Q(x)$ is the quotient obtained by dividing the denominator of $f(x)$ into the numerator and excluding the remainder.